# Risk Sensitive Investment Management
## Overview and Applications

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# Outline

# 1 - Introduction

**The Mean-Variance Revolution** - Modern Portfolio Theory, pioneered by Markowitz [9], Sharpe and Lintner, has produced a considerable impact on both financial theory and practice. Its tenets permeate the field of financial economics and form the foundation of the investment decision-making process.

*Limitations:*

- extreme sensitivity to parameter estimate;
- it is a one period model.

**The Merton Evolution** - The natural next step is to construct a multiperiod model.

Merton [10] formulates the investment problem as a stochastic control problem where the objective is to maximize the investor's utility. This model constitutes one of the few nonlinear stochastic control problem with an analytical solution.

*Limitations:*

- high degree of mathematical stylization;
- "curse of dimensionality".

**(Fractional) Kelly Criterion Investing** - Kelly criterion holds a special place in investment theory and practice: Ziemba [12] demonstrated that several great investors (Keynes, Buffett...) are Kelly criterion investors.

To address the riskiness of Kelly investing, Ziemba proposes fractional Kelly strategies: invest a proportion of the wealth in the Kelly portfolio and the remainder in the short-term rate.

*Limitations:*

- Kelly criterion investing is not for the faint-hearted;
- Fractional Kelly is not generally optimal (notable exception: asset prices are lognormally distributed);

Ideally, we would like to find an approach which combines

- the insights of Mean-Variance optimization;
- the continuous time approach of stochastic control;
- the intuition of fractional Kelly.

... and which would also be

- consistent with utility maximization;
- less susceptible to fall victim of the “curse of dimensionality”.

As we will see shortly, the risk-sensitive asset management model meets these requirements.

# 2 - What is Risk-Sensitive Control?

**A First Definition** - Risk-sensitive control is most simply defined as

- a generalization of classical stochastic control;
- in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization.

**The Risk-Sensitive Criterion** - In risk-sensitive control, the decision maker's objective is to select a control policy $h(t)$ to maximize the criterion

$$J(t, x, h; \theta) := -\frac{1}{\theta} \ln \mathbf{E} \left[ e^{-\theta F(t,x,h)} \right] \tag{1}$$

where

- $t$ and $x$ are the time and the state variable;
- $F$ is a given reward function;
- the risk sensitivity $\theta \in ]-1, 0[ \cup ]0, \infty)$ represents the decision maker's degree of risk aversion.

A Taylor expansion of the previous expression around $\theta = 0$ evidences the vital role played by the risk sensitivity parameter:

$$J(x, t, h; \theta) = \mathbf{E}\left[F(t, x, h)\right] - \frac{\theta}{2}\mathbf{Var}\left[F(t, x, h)\right] + O(\theta^2) \qquad (2)$$

- $\theta \to 0$, "risk-null": corresponds to classical stochastic control;
- $\theta < 0$: "risk-seeking" case corresponding to a maximization of the expectation of a convex decreasing function of $F(t, x, h)$;
- $\theta > 0$: "risk-averse" case corresponding to a minimization of the expectation of a convex increasing function of $F(t, x, h)$.

Therefore, risk-sensitive control differs from traditional stochastic control in that it explicitly models the risk-aversion of the decision maker as an integral part of the control framework, rather than importing it in the problem via an externally defined utility function.

# 3 - Asset Management Applications and Implications

**Bielecki and Pliska** [2] pioneered the application of risk-sensitive control to asset management.

They proposed that the reward function be defined as the logarithm of the investor's wealth $V$, i.e.

$$F(t, x, h) = \ln V(t, x, h)$$

*Interpretation*: the investor's objective is to maximize the risk-sensitive (log) return of the his/her portfolio.

**Risk-Sensitive Asset Management and utility maximization** - with this choice of reward function, we can express the control criterion as

$$J(t, x, h; \theta) := -\frac{1}{\theta} \ln \mathbf{E}\left[e^{-\theta \ln V(t,x,h)}\right] \tag{3}$$

Now, the expectation

$$\mathbf{E}\left[e^{-\theta \ln V(t,x,h)}\right] = \mathbf{E}\left[V(t, x, h)^{-\theta}\right] =: U_\theta(V_t) \tag{4}$$

can be interpreted as the expected utility of time $t$ wealth under the power utility (HARA) function.

**Risk-Sensitive Asset Management, Mean-Variance Analysis and Kelly criterion** - the Taylor expansion becomes

$$J(t,x;\theta) = \mathbf{E}\left[\ln V(t,x,h)\right] - \frac{\theta}{2}\mathbf{Var}\left[\ln V(t,x,h)\right] + O(\theta^2) \qquad (5)$$

Ignoring higher order terms, we recover the mean-variance optimization criterion...

... and the Kelly criterion portfolio in the limit as $\theta \to 0$.

# 4 - The Diffusion Risk-Sensitive Asset Management

**The Diffusion Model** - Embedding the investor's risk-sensitivity in the control criterion gives us more leeway in the specification of the asset market than would be obtained in the Merton approach.

In particular, Bielecki and Pliska [2] propose a factor model in which the prices of the risky assets follow a SDE of the form

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= (a + AX(t))_i dt + \sum_{k=1}^{n+m} \sigma_{ik} dW_k(t) \\ & S_i(0) = s_i, \quad i = 1, \ldots, m \end{aligned} \tag{6}$$

where $W(t)$ is a $(n+m)$-dimensional Brownian motion.

We also consider a short-term rate process satisfying

$$\frac{dS_0(t)}{S_0(t)} = \left(a_0 + A_0' X(t)\right) dt, \qquad S_0(0) = s_0 \tag{7}$$

The asset prices drift depends on $n$ valuation factors modelled as affine processes with constant diffusion

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{8}$$

These factors must be specified, but they could include macroeconomic, microeconomic or statistical variables.

We make only two minor assumptions:

**Assumption**

$$\Lambda\Lambda' > 0$$
$$\Sigma\Sigma' > 0$$

Under these conditions, the logarithm of the investor's wealth is given by the SDE

$$\begin{aligned}\ln V(t) &= \ln v + \int_0^t \left(a_0 + A_0' X(s)\right) + h(s)' \left(\hat{a} + \hat{A} X(s)\right) ds \\ &\quad -\frac{1}{2}\int_0^t h(s)'\Sigma\Sigma' h(s) ds + \int_0^t h(s)'\Sigma dW(s), \end{aligned} \tag{9}$$

where $V(0) = v$, $h$ is the $m$-dimensional vector of portfolio weights and we used the notation $\hat{a} := a - a_0\mathbf{1}$ and $\hat{A} := A - \mathbf{1}A_0'$.

We immediately notice that the equation for $V$ solely depends on the valuation factors (the state process): it is independent from the asset prices.

The next step is due to Kuroda and Nagai [8] who ingeniously observed that under an appropriately chosen change of measure, the risk-sensitive criterion can be expressed as

$$I(v, x; h; t, T) = \ln v - \frac{1}{\theta} \ln \mathbf{E}^{\theta} \left[ \exp \left\{ \theta \int_t^T g(X_s, h(s); \theta) ds \right\} \right] \tag{10}$$

where

$$g(x, h; \theta) \;\; = \;\; \frac{1}{2} (\theta + 1) \, h' \Sigma \Sigma' h - a_0 - A_0' x - h'(\hat{a} + \hat{A} x) \tag{11}$$

and where the factor dynamics under the new measure $\mathbb{P}_h^{\theta}$ is given by:

$$dX_s = \left( b + BX_s - \theta \Lambda \Sigma' h(s) \right) ds + \Lambda dW_s^{\theta} \tag{12}$$

In this formulation, the problem is a standard Linear Exponential-of-Quadratic Gaussian (LEQG) control problem which can be solved exactly (up to the resolution of a system of Riccati equations).

**Special Case: the factors and assets are uncorrelated** - In the case when $\Lambda\Sigma' = 0$, security risk and factor risk are uncorrelated, the evolution of $X_t$ under the measure $\mathbb{P}_h^\theta$ given in equation (12) can be expressed as:

$$dX_s = (b + BX_s)\, ds + \Lambda dW_s^\theta$$

The evolution of the state is therefore independent of the control variable $h$ and, as a result, the control problem can be solved through a pointwise maximisation of the auxiliary criterion function $I(v, x; h; t, T)$.

The optimal control $h^*$, in this case, is the maximizer of the function $g(x; h; t, T)$ given by

$$h^* = \frac{1}{\theta + 1}(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}x\right)$$

which is a position of $\frac{1}{\theta+1}$ in the Kelly portfolio.

Let $\Phi(t,x)$ be the value function corresponding to the exponential of integral criterion $I(v,x;h;t,T)$. Substituting the value of $h^*$ in the equation for $g$, we note that

$$\begin{aligned}
\Phi(t,x) &= \sup_{h\in\mathcal{A}(T)} I(v,x;h;t,T) \\
&= -\frac{1}{\theta}\ln \mathbf{E}^{\theta}\left[\exp\left\{\theta\int_0^{T-t} g(x,h^*(s);t,T;\theta)ds\right\} v^{-\theta}\right]
\end{aligned}$$

The PDE for $\Phi$ can now be obtained directly via an exponential transformation and Feynman-Kac.

**In the general case,** the value function $\Phi$ for the auxiliary criterion function $I(v, x; h; t, T)$, defined as

$$\Phi(t, x) = \sup_{\mathcal{A}(T)} I(v, x; h; t, T)$$

satisfies the Hamilton-Jacobi-Bellman Partial Differential Equation (HJB PDE)

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L_t^h \Phi(X(t)) = 0 \qquad (13)$$

where

$$\begin{aligned} L_t^h \Phi(t, x) &= \left(b + Bx - \theta \Lambda \Sigma' h(s)\right)' D\Phi + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi\right) \\ &\quad - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta) \end{aligned}$$

and subject to terminal condition $\Phi(T, x) = \ln v$.

Solving the optimization problem, we find that the optimal investment policy $h^*(t)$ is given by

$$h^*(t) = \frac{1}{\theta + 1} \left( \Sigma \Sigma' \right)^{-1} \left[ \hat{a} + \hat{A} X(t) - \theta \Sigma \Lambda' D \Phi(t, X(t)) \right]$$

The solution of the PDE is of the form

$$\Phi(t,x) = x'Q(t)x + x'q(t) + k(t)$$

where $Q(t)$ solves a $n$-dimensional matrix Riccati equation and $q(t)$ solves a $n$-dimensional linear ordinary differential equation depending on $Q$.

**Dimensionality** - the effective dimension of the risk-sensitive asset management model is the number of factors rather than the number of assets.

The limited impact of the number of assets is particularly important since for practical applications we would typically use only a few factors (possibly 3 to 5) to parametrize a large cohort of assets and asset classes (possibly several dozens).

The risk-sensitive asset management model is therefore particularly efficient from a computational perspective.

**Theorem (Mutual Fund Theorem (Davis and Lleo [6]))**

*Any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^C(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned} \tag{14}$$

*and respective allocation to the money market account given by:*

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned}$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$.*

The allocation between the two funds is a sole function of the investors's risk sensitivity $\theta$.

As $\theta \to 0$, the investor's wealth gets invested in the Kelly criterion portfolio (portfolio K).

As $\theta \to \infty$, the investor's wealth gets invested in portfolio C. The investment strategy of this portfolio can be interpreted as a large position in the short-term rate and a set of positions trading on the comovement of assets and valuation factors.

Note that when we assume that there are no underlying valuation factors, the risky securities follow geometric Brownian motions with drift vector $\mu$ and the money market account becomes the risk-free asset (i.e. $a_0 = r$ and $A_0 = 0$).

In this case $\Sigma\Lambda' = 0$ and we can then easily see that fund $C$ is fully invested in the risk-free asset.

As a result, we recover Merton's Mutual Fund Theorem for $m$ risky assets and a risk-free asset.

**Fractional Kelly Strategies** - we now propose to adapt the concept of fractional Kelly strategy to our factor model and to the findings expressed in the mutual fund theorem.

Instead of regarding the fractional Kelly strategy as a split between the Kelly portfolio and the short-term rate, we propose to define it as a split between the Kelly portfolio and the portfolio C defined in the mutual fund theorem.

This redefinition has two important consequences

- the fractional Kelly portfolios are always optimal portfolios;
- in the lognormal case, our adapted definition reverts to the 'classical' definition;

**Asset Classes** - The risk-sensitive asset management model was originally set in an environment of equity-like securities exhibiting a geometric growth.

Bielecki and Pliska [3] sucessfully used an approach due to Rutkowski [11] to include non-defaultable zero-coupon bonds in the risk-sensitive asset management model. Rutkowski's key insight is that by interpreting yields as underlying factors, one can then model the price of fixed income securities using the same type of geometric dynamics as used to model asset prices in the risk-sensitive asset management model.

As a result, Rutkowski's approach demonstrates that in the risk-sensitive asset management model, the modelling of the actual securities can be undertaken separately from the design of the control model.

# 5 - Beyond Asset Allocation (Part I): Managing to Benchmarks and to Liabilities

The risk sensitive asset management model can be extended to take into account benchmarks and liabilities without losing its elegance, and as importantly, without increasing the level of complexity of the approach.

**Benchmarked Asset Management** - in this case, the investor selects an asset allocation to outperform a given investment benchmark.

Davis and Lleo [6] propose that the reward function $F(t, x; h)$ be defined as the (log) excess return of the investor's portfolio over the return of the benchmark, i.e.

$$F(t, x, h) := \ln \frac{V(t, x, h)}{L(t, x, h)}$$

where $L$ is the level of the benchmark.

Furthermore, the dynamics of the benchmark is modelled by the SDE:

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t), \qquad L(0) = l \qquad (15)$$

This formulation is wide enough to encompass a multitude of situations such as:

- *the single benchmark case*, where the benchmark is, for example, an equity index such as the S&P500 or the FTSE.
- *the single benchmark plus alpha*, where, for example, a hedge fund has for benchmark a target based on a short-term interest rate plus alpha.
- *the composite benchmark case*. for example a benchmark constituted of 5% cash, 35% Citigroup World Government Bond Index, 25% S&P 500 and 35% MSCI EAFE.
- *the composite benchmark plus alpha*, a combination of the previous two cases.

By Itô's lemma, the log of excess return in response to a strategy $h$ is

$$\begin{aligned} F(t,x;h) &= \ln\frac{v}{l} + \int_0^t d\ln V(s) - \int_0^t d\ln L(s) \\ &= \ln\frac{v}{l} + \int_0^t \left(a_0 + A_0' X(s) + h(s)'\left(\hat{a} + \hat{A}X(s)\right)\right) ds \\ & \quad -\frac{1}{2}\int_0^t h(s)'\Sigma\Sigma' h(s) ds + \int_0^t h(s)'\Sigma dW(s) \\ & \quad -\int_0^t (c + C'X(s)) ds + \frac{1}{2}\int_0^t \varsigma'\varsigma ds \\ & \quad -\int_0^t \varsigma' dW(s) \end{aligned} \tag{16}$$

$$F(0,x;h) = f_0 := \ln\frac{v}{l}$$

Following an appropriate change of measure, the criterion function can be expressed as

$$I(f_0, x; h; t, T) = \ln f_0 - \frac{1}{\theta} \ln \mathbf{E}^{\theta} \left[ \exp \left\{ \theta \int_0^{T-t} g(X_s, h(s); \theta) ds \right\} \right]$$

where

$$\begin{aligned} g(x, h; \theta) &= \frac{1}{2} (\theta + 1) h' \Sigma \Sigma' h - a_0 - A_0' x - h'(\hat{a} + \hat{A} x) \\ &\quad - \frac{1}{2} \theta (h' \Sigma \varsigma + \varsigma' \Sigma' h) + (c + C' x) + \frac{1}{2} (\theta - 1) \varsigma' \varsigma \end{aligned}$$

Once again, our control problem simplifies into a LEQG problem.

Let $\Phi$ be the value function for the auxiliary criterion function $I(f_0, x; h; t, T)$. Then $\Phi$ is defined as

$$\Phi(t, x) = \sup_{\mathcal{A}(T-t)} I(v, l, x; h; t, T)$$

and it satisfies the HJB PDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in \mathbb{R}^m} L_t^h \Phi = 0 \tag{17}$$

where

$$\begin{aligned} L_t^h \Phi &= \left(b + Bx - \theta \Lambda (\Sigma' h - \varsigma)\right)' D\Phi + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda' D^2 \Phi\right) \\ & \quad - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta) \end{aligned} \tag{18}$$

Solving the optimization problem, we find that the optimal investment policy $h^*(t)$ is given by

$$h^*(t) = \frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}x - \theta\Sigma\Lambda' D\Phi + \theta\Sigma\varsigma\right) \tag{19}$$

The solution of the PDE is still of the form

$$\Phi(t,x) = x'Q(t)x + x'q(t) + k(t)$$

where $Q(t)$ solves a $n$-dimensional matrix Riccati equation and $q(t)$ solves a $n$-dimensional linear ordinary differential equation.
In addition, we have a new mutual fund theorem...

## Theorem (Benchmarked Fund Theorem (Davis and Lleo [6]))

*Given a time $t$ and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^C(t) &= (\Sigma\Sigma')^{-1}\left[\Sigma\varsigma - \Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)\right] \end{aligned} \quad (20)$$

*and respective allocation to the money market account given by:*

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left[\Sigma\varsigma - \Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)\right] \end{aligned}$$

*Moreover, if an investor has a risk sensitivity $\theta$, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$.*

*Role of $\theta$ in the benchmark case* - while in the asset only case $\theta$ represents the sensitivity of an investor to total risk, in the benchmark case, $\theta$ seems rather to represent the investor's sensitivity to active risk.

When $\theta$ is low, the investor will take more active risk by investing larger amounts into the log-utility portfolio.

On the other hand, when $\theta$ is high, the investor will divert most of his/her funds to the correction fund, which is dominated by the term $(\Sigma\Sigma')^{-1}\Sigma\varsigma$, a term designed to track the index.

Hence, in the benchmark case, the investor already takes the benchmark risk as granted. The main unknown is therefore how much additional risk the investor is willing to take in order to outperform the benchmark.

**Asset and Liability Management** - Davis and Lleo [5] propose a model in which the investor's position is funded by a liability whose dynamics follows the same type of SDE as the benchmark considered in the previous problem

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t), \qquad L(0) = l \tag{21}$$

The objective of the investor is to maximize the risk-sensitive (log) return-on-equity of the portfolio, where the equity is defined as

$$E(t) := V(t) - L(t)$$

with

$$E(0) = e_0 := v - l > 0$$

To achieve this objective, the investor can

- change the allocation of the asset portfolio;
- increase of decrease leverage by either paying down a part of the liability or issuing more liability.

As a result, we now consider an additional control variable: the leverage ratio, $\rho$, that we define as

$$\rho(t) := \frac{V(t)}{E(t)} \tag{22}$$

and the dynamics of the equity can be expressed as

$$\begin{aligned} & \frac{dE(t)}{E(t)} \\ = \quad & (c + C'X(t))dt \\ & +\rho(t)\left[\left(a_0 + A_0'X(t)\right) + h'(t)\left(\hat{a} + \hat{A}X(t)\right)\right. \\ & \left.-(c + C'X(t))\right] dt + \left[\rho(t)(h'(t)\Sigma - \varsigma') + \varsigma'\right] dW(t), \\ & E(0) = e_0 \end{aligned} \tag{23}$$

Although the control problem is now more complicated, it can still be solved analytically using the same technique as the asset-only case and the benchmark case.

In particular, the optimal asset allocation is given by

$$h^*(t) = \frac{1}{\rho^*(t)} \frac{1}{\theta + 1} (\Sigma\Sigma')^{-1} \left( \hat{a} + \hat{A}x - \frac{\theta}{2} \Sigma \Lambda' D \Phi + (\theta + 1)(\rho - 1) \Sigma \varsigma \right) \tag{24}$$

where we note that the investment in the Kelly portfolio has been 'de-levered'.

## 6 - Beyond Asset Allocation (Part II): Jumps!

Our current research concerns the inclusion of credit risk and of a credit asset class to extend risk-sensitive asset management beyond the realm of stock-like securities and of non-defaultable bonds.

This aim has motivated an extension of the modelling framework beyond diffusion processes and into jump-diffusion processes.

**Asset and Factor Modelling** - In a jump diffusion setting, the factor dynamics and asset prices are respectively given by

$$dX(t) = (b + BX(t^-))dt + \Lambda dW(t) + \int_{\mathbf{Z}} \xi(z)\bar{N}_{\mathbf{p}}(dt, dz), \qquad X(0) = x \tag{25}$$

and

$$\begin{aligned} \frac{dS_i(t)}{S_i(t^-)} &= (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \gamma_i(z)\bar{N}_{\mathbf{p}}(dt, dz), \\ & S_i(0) = s_i, \quad i = 1, \ldots, m \end{aligned} \tag{26}$$

For notational convenience, we define a Poisson random measure $\bar{N}_{\mathbf{p}}(dt, dz)$ as

$$\bar{N}_{\mathbf{p}}(dt, dz) = \begin{cases} N_{\mathbf{p}}(dt, dz) - \nu(dz)dt =: \tilde{N}_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z}_0 \\ N_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z} \backslash \mathbf{Z}_0 \end{cases}$$

where the measure $\nu$ is the compensator of the Poisson random measure.

The class of Poisson random measures, which we use in our approach, gives us the flexibility to model a wide range of jump-diffusion specifications, including the class of Lévy processes (see for example Ikeda and Watanabe [7]).

Following our earlier line of reasoning, in the asset-only case the value function $\Phi$ satisfies the integro-differential HJB PDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in J} L_t^h \Phi(X(t)) = 0 \tag{27}$$

where

$$\begin{aligned} L_t^h \Phi(t,x) &= \Big( b + Bx - \theta \Lambda \Sigma' h(s) \\ &\quad + \int_{\mathbf{Z}} \xi(z) \left[ \left(1 + h'\gamma(z)\right)^{-\theta} - \mathit{1}_{\{z \in \mathbf{Z}_0\}} \right] \nu(dz) \Big)' D\Phi \\ &\quad + \frac{1}{2} \mathrm{tr}\left(\Lambda \Lambda' D^2 \Phi\right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h; \theta) \\ &\quad + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left( e^{-\theta(\Phi(t, x + \xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) \end{aligned}$$

and $g$ is the instantaneous reward function. The terminal condition is $\Phi(T, x) = \ln v$.

**Existence and Uniqueness of the Optimal Control** - we start by developing the supremum:

$$
\begin{aligned}
& \sup_{h\in J} L_t^h \Phi \\
= \; & (b+Bx)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi + a_0 + A_0' x \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left( e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1\right) - \xi'(z) D\Phi \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) \\
& + \sup_{h\in J} \left\{ -\frac{1}{2}(\theta+1) h'\Sigma\Sigma' h - \theta h'\Sigma\Lambda' D\Phi + h'(\hat{a} + \hat{A}x) \right. \\
& - \frac{1}{\theta}\int_{\mathbf{Z}} \left\{ \left(1-\theta\xi'(z)D\Phi\right)\left[\left(1+h'\gamma(z)\right)^{-\theta} - 1\right] \right. \\
& \left. \left. + \theta h'\gamma(z)\mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right\}
\end{aligned}
\tag{28}
$$

Next, notice that the term

$$-\frac{1}{2}(\theta+1)\,h'\Sigma\Sigma' h - \theta h'\Sigma\Lambda' D\Phi + h'(\hat{a} + \hat{A}x) - \int_{\mathbf{Z}} h'\gamma(z) 1_{\mathbf{Z}_0}(z)\nu(dz)$$

is strictly concave in $h$ $\forall z \in \mathbb{Z}$ a.s. $d\nu$.

If we assume that the assets and factors do not have simultaneous jumps, i.e $\forall z \in \mathbf{Z}$, $\gamma(z)\xi'(z) = 0$, the nonlinear jump-related term

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left(1-\theta\xi'(z)D\Phi\right)\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]\right\}\nu(dz)$$

simplifies to

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]\right\}\nu(dz)$$

which is also concave in $h$ $\forall z \in \mathbb{Z}$ a.s. $d\nu$.

Therefore, the supremum is reached for a unique optimal control $h^*$, which is an interior point of the set $J$, and the supremum, evaluated at $h^*$, is finite.

**Special Case: factors as diffusion** - if we model the factor dynamics as diffusion processes and only add jumps to the asset prices, then our control problem remains, broadly speaking, a diffusion problem.

We conjecture that in this case the HJB equation has a classical $C^{1,2}$ solution...

*... to be continued!...*

**Viscosity Solutions** - In the general case, however, we can no longer find either an analytical solution or even a classical solution.

Nevertheless, a weak-sense solution, specifically a viscosity solution, exists. The theory of viscosity solutions represents a powerful set of techniques to solve a wide range of elliptical and parabolic PDEs (see [4] for more details).

In the risk-sensitive asset management case, we have proved that the value function $\Phi$ is the unique, continuous viscosity solution of the HJB PDE satisfying appropriate growth condition.

**Numerical implementation** - Our interest in viscosity solutions is also practical.

Indeed, Barles and Souganidis [1]' 'stability result' can be used to prove convergence of a wide range of numerical schemes to the viscosity solution of a PDE and therefore establishes a strong connection between the theory of viscosity solution and numerical analysis.

And, since the effective dimension of our control problem remains equal to the number of factors rather than assets, we believe that reasonably practical asset allocation problems can be solved.

**Benchmark and ALM** - the benchmarked asset allocation and the ALM problems can be extended to the jump diffusion in a similar way and with similar conclusions: the value function $\Phi$ is the unique, continuous viscosity solution of the problem's HJB PDE.

# Further Research

Currently, our main research question relates to the modelling of credit securities (defaultable zero-coupon bonds and CDS) for inclusion in the investment universe.

Modelling credit securities would enable us to solve investment problems across three asset classes: equity, fixed income and credit products.

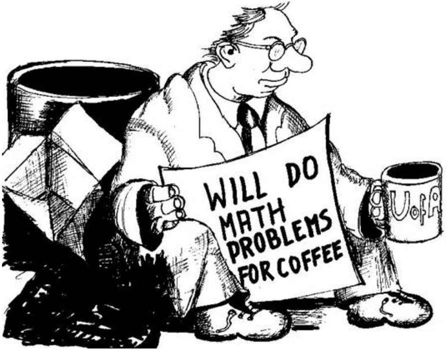
*... to be continued!...*

# To conclude,

Risk-sensitive control

- provides a promising setting for a wide range of investment management problems;
- combines the mathematical elegance of the Merton model with the insights of Mean-Variance optimization and the intuition of fractional Kelly;
- is numerically appealing, since the effective dimensionality is the number of factors rather than the number of assets;
- has closed-formed solutions in the diffusion case and can be solved numerically in the jump-diffusion case.

**Thank you!**




[1]

**Any question?**

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